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Authors
Farouki, RT
Sakkalis, T

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Construction of rational curves with rational arc lengths by direct integration

Rida T. Farouki
Department of Mechanical and Aerospace Engineering,
University of California, Davis, CA 95616, USA

Takis Sakkalis
Mathematics Laboratory, Agricultural University of Athens,
75 Iera Odos, Athens 11855, GREECE

Abstract

A methodology for the construction of rational curves with rational arc length functions, by direct integration of hodographs, is developed. For a hodograph of the form $\mathbf{r}'(\xi) = (u^2(\xi) - v^2(\xi), 2\,u(\xi)v(\xi))/w^2(\xi)$, where $w(\xi)$ is a monic polynomial defined by prescribed simple roots, we identify conditions on the polynomials $u(\xi)$ and $v(\xi)$ which ensure that integration of $\mathbf{r}'(\xi)$ produces a rational curve with a rational arc length function $s(\xi)$. The method is illustrated by computed examples, and a generalization to spatial rational curves is also briefly discussed. The results are also compared to existing theory, based upon the dual form of rational Pythagorean–hodograph curves, and it is shown that direct integration produces simple low–degree curves which otherwise require a symbolic factorization to identify and cancel common factors among the curve homogeneous coordinates.

Keywords: rational curves; arc length function; Pythagorean–hodograph curves; points at infinity; polynomial roots; rational function integration; residues.

e–mail: farouki@ucdavis.edu, psakkalis@aua.gr

1 Introduction

The distinctive feature of a polynomial or rational Pythagorean–hodograph (PH) curve $\mathbf{r}(\xi)$ is that the parametric speed $\sigma(\xi) = |\mathbf{r}'(\xi)|$, which specifies the rate of change $\mathrm{d}s/\mathrm{d}\xi$ of the arc length s with respect to the parameter ξ, is a polynomial or rational function [2] of ξ. Polynomial PH curves may be constructed by integrating a polynomial hodograph $\mathbf{r}'(\xi)$ whose components satisfy a Pythagorean condition. However, an alternative approach is usually invoked [9] for rational PH curves, since integration of a rational hodograph does not necessarily generate a rational curve. A further consequence is that, whereas all polynomial PH curves have polynomial arc length functions, only a subset of the rational PH curves admits rational arc lengths.

This important distinction between polynomial and rational PH curves is often overlooked. For example, it has been stated [8] that "The Pythagorean–hodograph (PH) curves ... are a special class of polynomial/rational curves with polynomial/rational speed functions. They have polynomial/rational arc lengths ..." As demonstrated in [3], however, only a special instance[1] of the curves considered in [8] admits a rational arc length.

The focus of the present study is to develop a characterization of rational curves with rational arc lengths, by an approach different from that employed in [9] — namely, by investigating the conditions under which certain types of rational functions have rational integrals. Specifically, in the planar case, for a given monic polynomial $w(\xi)$ with simple roots we identify the conditions on polynomials $u(\xi), v(\xi)$ with $\gcd(u, v) = \gcd(u, w) = \gcd(v, w) = 1$ such that the rational functions u^2/w^2, uv/w^2, v^2/w^2 all admit rational integrals. This formulation is then generalized to rational space curves.

The remainder of this paper is organized as follows. Section 2 formulates the problem of identifying rational plane curves that have rational arc lengths in terms of the indefinite integrals of u^2/w^2, uv/w^2, v^2/w^2. Assuming that $\gcd(u, v) = 1$ and w has prescribed simple roots — which specify the points at infinity of the curve $\mathbf{r}(\xi)$ — Section 3 then identifies linear constraints on u and v which guarantee that these integrals are rational. By expressing these constraints in terms of the residues of the integrands at their poles, Section 4 facilitates analysis of the linear system embodying them, and of the existence and construction of solutions. In Section 5, this direct integration approach

[1] In fact, this special case corresponds to a polynomial PH curve subject to a fractional linear (Möbius) parameter transformation.

is compared with the characterization in [9] of rational curves with rational arc lengths as the evolutes of general rational PH curves, and examples are employed to show that it can provide simpler and more direct constructions of low–degree curves, avoiding the need to identify any common factors among the homogeneous coordinates. Finally, Section 6 briefly sketches an extension of the method to rational space curves, while Section 7 summarizes the key results of the present study and identifies further avenues of investigation.

2 Construction by direct integration

We are interested in developing methods to identify when rational PH curves possess rational arc lengths, by the direct integration of their hodographs. We focus initially on plane curves $\mathbf{r}(\xi) = (x(\xi), y(\xi)) = (X(\xi)/W(\xi), Y(\xi)/W(\xi))$ specified by homogeneous coordinate polynomials $W(\xi), X(\xi), Y(\xi)$ and then discuss a generalization to spatial curves in Section 6.

Consider the rational hodograph $\mathbf{r}'(\xi) = (x'(\xi), y'(\xi))$ defined by

$$x'(\xi) \,=\, \frac{e(\xi)}{d(\xi)}\,, \quad y'(\xi) \,=\, \frac{f(\xi)}{d(\xi)}\,, \tag{1}$$

for polynomials $d(\xi), e(\xi), f(\xi)$ satisfying $\gcd(e, d) = 1$ and/or $\gcd(f, d) = 1$. If $x(\xi)$ and $y(\xi)$ are to be rational functions, $d(\xi)$ cannot possess any simple roots, since they will incur transcendental terms upon integrating (1). We consider here the simplest (and most general) assumption that satisfies this requirement — namely, that $d(\xi)$ is the perfect square of a polynomial $w(\xi)$ with only simple roots, i.e., $d(\xi) = w^2(\xi)$. For the parametric speed

$$\sigma(\xi) \,=\, \frac{\mathrm{d}s}{\mathrm{d}\xi}$$

of $\mathbf{r}(\xi)$ to be rational, $e^2(\xi) + f^2(\xi)$ must also be a perfect square, and this implies [7] that[2]

$$e(\xi) \,=\, u^2(\xi) - v^2(\xi)\,, \quad f(\xi) \,=\, 2\,u(\xi)v(\xi)$$

for relatively prime polynomials $u(\xi)$ and $v(\xi)$. Consequently, $e^2(\xi)+f^2(\xi) = [\,u^2(\xi) + v^2(\xi)\,]^2$, and we have

$$x' \,=\, \frac{u^2 - v^2}{w^2}\,, \quad y' \,=\, \frac{2\,uv}{w^2}\,, \quad s' \,=\, \frac{u^2 + v^2}{w^2}\,. \tag{2}$$

[2]We focus here on *primitive* hodographs, satisfying $\gcd(e, f) = 1$ when $\gcd(u, v) = 1$.

Hence, the integrals

$$x = \int \frac{u^2 - v^2}{w^2}\,\mathrm{d}\xi\,, \quad y = \int \frac{2\,uv}{w^2}\,\mathrm{d}\xi\,, \quad s = \int \frac{u^2 + v^2}{w^2}\,\mathrm{d}\xi \tag{3}$$

must reduce to rational expressions in ξ for $\mathbf{r}(\xi)$ to be a rational curve with a rational arc length. This is equivalent to requiring that the simpler integrals

$$\int \frac{u^2}{w^2}\,\mathrm{d}\xi\,, \quad \int \frac{uv}{w^2}\,\mathrm{d}\xi\,, \quad \int \frac{v^2}{w^2}\,\mathrm{d}\xi \tag{4}$$

yield rational expressions. Our goal is to identify, for a given $w(\xi)$ with only simple roots, conditions on $u(\xi), v(\xi)$ which ensure that the integrals (4) are rational. We assume, without loss of generality, that $w(\xi)$ is monic. We also assume that $\gcd(u,w) = \gcd(v,w) = 1$, to eliminate the possibility of factors common to the numerators and denominators of the integrands in (4).

When such conditions are satisfied, it is understood that any non–constant factors common to $W(\xi), X(\xi), Y(\xi)$ in the rational curve $x(\xi) = X(\xi)/W(\xi)$, $y(\xi) = Y(\xi)/W(\xi)$ obtained from (3) must be cancelled out, so as to ensure that $\gcd(W,X,Y) = 1$. Note that the roots of $W(\xi)$, which identify points at infinity of the curve, are the same as the roots of $w(\xi)$, but their multiplicities may differ. To verify this, we argue by contradiction. With $\gcd(W,X,Y) = 1$ and $\gcd(u,v) = \gcd(u,w) = \gcd(v,w) = 1$, we have

$$(u^2 - v^2)\,W^2 = w^2(WX' - W'X) \quad \text{and} \quad 2\,uv\,W^2 = w^2(WY' - W'Y)\,.$$

Since $\gcd(u,w) = \gcd(v,w) = 1$, the second equation implies that every root of w is also a root of W. Now suppose ξ_* is a root of W of multiplicity k, that is *not* a root of w. Then $(\xi - \xi_*)^k$ must divide $W'X$ and $W'Y$, and since $(\xi - \xi_*)^{k-1}$ divides W', we infer that $(\xi - \xi_*)$ must divide X and Y. But this contradicts $\gcd(W,X,Y) = 1$.

Remark 1. For any $w(\xi) \in \mathbb{R}[\xi]$, the set $S = \{\, u(\xi), v(\xi) \in \mathbb{R}[\xi] \,\}$ of pairs of polynomials for which the integrals (4) are rational is closed under the map $(u(\xi), v(\xi)) \to (\alpha\, u(\xi) + \beta\, v(\xi), \gamma\, u(\xi) + \delta\, v(\xi))$ with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

3 Specifying the points at infinity

The main problem addressed herein may be phrased as follows.

Problem 1. *For any specified polynomial $w(\xi) \in \mathbb{R}[\xi]$ with simple roots, what conditions on the polynomials $u(\xi)$ and $v(\xi)$ ensure that the three indefinite integrals* (4) *are all rational functions of ξ?*

To address this problem, we need to examine the root structure of $w(\xi)$. These roots identify the *poles* of the rational integrands in (3), and the *points at infinity* of the rational curve $\mathbf{r}(\xi) = (X(\xi)/W(\xi), Y(\xi)/W(\xi))$. In order for a rational function to possess a rational indefinite integral, the residues at its poles must vanish, since they will incur transcendental (trigonometric or logarithmic) terms if non–zero. Now if $r(\xi)$ is any of the rational integrands in (4) and ξ_* is a pole of $r(\xi)$ of multiplicity m, then the residue of $r(\xi)$ at ξ_* is defined [5] as

$$\operatorname*{residue}_{\xi=\xi_*} r(\xi) \,=\, \frac{1}{(m-1)!} \left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}\xi^{m-1}} (\xi - \xi_*)^m r(\xi) \right]_{\xi=\xi_*} . \tag{5}$$

Equation (5) holds whether the degree of the numerator of $r(\xi)$ is less than, equal to, or greater than that of the denominator. Now if the polynomial $w(\xi)$ in (4) is of degree n, it may possess n simple roots, a single root of multiplicity n, or a combination of simple and multiple roots whose multiplicities sum to n. We focus here on the generic case of n simple roots, but first we mention the case of a single root ξ_* of multiplicity n, so that $w(\xi) = (\xi - \xi_*)^n$.

A polynomial curve $\mathbf{r}(t)$ has a single point at infinity, of multiplicity equal to the curve degree, that corresponds to an infinite value of the parameter t. Under the Möbius (fractional linear) parameter transformation $t \to \xi$ defined for $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta\,\xi_*$ by

$$\xi \,=\, \frac{\xi_* t + \alpha}{t + \beta} \,, \tag{6}$$

the value $t = \infty$ is mapped to $\xi = \xi_*$. Thus, in the case $w(\xi) = (\xi - \xi_*)^n$, on substituting (6) into the curve defined by (3) and clearing denominators, we obtain a *polynomial* curve. Hence, any $u(\xi), v(\xi)$ that make the integrals (4) rational with $w(\xi) = (\xi - \xi_*)^n$ is actually just a polynomial PH curve $\mathbf{r}(t)$, parameterized by the variable t defined by the inverse of (6), namely

$$t \,=\, \frac{\alpha - \beta\,\xi}{\xi - \xi_*} \,.$$

We assume henceforth that $w(\xi)$ has n simple (real or complex conjugate) roots $\xi_1,\ldots,\xi_n$. Assuming that it is monic, we may write

$$w(\xi) \,=\, \prod_{j=1}^{n}(\xi-\xi_j)\,, \tag{7}$$

and it is convenient to introduce the notation

$$w_k(\xi) \,=\, \prod_{\substack{j=1\\ j\neq k}}^{n}(\xi-\xi_j)\,, \quad k=1,\ldots,n\,,$$

i.e., $w_k(\xi)$ omits the factor $\xi-\xi_k$ from $w(\xi)$, so $w_k(\xi_k)\neq 0$. To ensure *regular* curves $\mathbf{r}(\xi)$, that are free of cusps and satisfy $\mathbf{r}'(\xi)\neq\mathbf{0}$ for all ξ, we henceforth assume that $u(\xi),v(\xi)$ are relatively prime, i.e., $\gcd(u,v)=1$.

Let $r(\xi)$ denote any of the rational integrands in (4). Since each root ξ_k of $w(\xi)$ is a double pole of $r(\xi)$, the residues (5) at these poles are

$$\operatorname*{residue}_{\xi=\xi_k} r(\xi) \,=\, \left[\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi-\xi_k)^2 r(\xi)\right]_{\xi=\xi_k}, \quad k=1,\ldots,n\,.$$

Consequently, we have

$$\operatorname*{residue}_{\xi=\xi_k}\frac{u^2}{w^2} = \left[\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{u^2}{w_k^2}\right]_{\xi=\xi_k} \,=\, 2\left[\frac{w_k u'-w_k' u}{w_k^2}\frac{u}{w_k}\right]_{\xi=\xi_k},$$

and an analogous expression holds for v^2/w^2. Furthermore, in the case of the integrand uv/w^2, we have

$$\operatorname*{residue}_{\xi=\xi_k}\frac{uv}{w^2} \,=\, \left[\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{uv}{w_k^2}\right]_{\xi=\xi_k} = \left[\frac{w_k u'-w_k' u}{w_k^2}\frac{v}{w_k}+\frac{w_k v'-w_k' v}{w_k^2}\frac{u}{w_k}\right]_{\xi=\xi_k}.$$

Hence, for the integrals of u^2/v^2 and v^2/w^2 to be rational, we must have

$$(w_k u'-w_k' u)u \,=\, (w_k v'-w_k' v)v \,=\, 0 \tag{8}$$

at each root $\xi_1,\ldots,\xi_n$ of w, and for the integral of uv/w^2 to be rational we require

$$(w_k u'-w_k' u)v + (w_k v'-w_k' v)u \,=\, 0 \tag{9}$$

at $\xi_1,\ldots,\xi_n$. Thus, for all three of the integrals (4) to be rational under the assumption that $\gcd(u,v)=1$, i.e., u and v do not vanish simultaneously, we must have

$$w_k u' - w_k' u \;=\; w_k v' - w_k' v \;=\; 0 \qquad \text{for } \; \xi=\xi_1,\ldots,\xi_n\,. \tag{10}$$

From the conditions (10), we may deduce the following result.

Proposition 1. *Let $u(\xi),v(\xi),w(\xi)$ be such that w has only simple roots,* $\gcd(u,v)=1$*, and the integrals* (4) *are rational. Then w divides $uv'-u'v$.*

Proof : Let ξ_k be any root of w. Then by the preceding argument, at $\xi=\xi_k$ we have $w_k u' - w_k' u = w_k v' - w_k' v = 0$ or

$$\begin{bmatrix} u' & -u \\ v' & -v \end{bmatrix} \begin{bmatrix} w_k \\ w_k' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and since $w_k(\xi_k)\neq 0$ we must have $uv'-u'v=0$ at $\xi=\xi_k$. Together with the fact that w has distinct roots, this implies that w divides $uv'-u'v$. ∎

For a given polynomial $w(\xi)$ of degree n, the polynomials $u(\xi),v(\xi)$ must be appropriately chosen to ensure a rational arc length. If $w(\xi)$ is specified by its n roots $\xi_1,\ldots,\xi_n$, the satisfaction of (10) at each of these roots yields n linear homogeneous constraints on the coefficients of $u(\xi)$ and $v(\xi)$. Since this linear system is the same for $u(\xi)$ and $v(\xi)$, the matrix defining it must be rank deficient by at least 2 to obtain linearly–independent polynomials.

Remark 2. If the integrals (4) are rational when w has only simple roots, then $uv'-u'v = p\,w$ for some polynomial p. Generically, this implies that $2\,m \geq n+2$ if $\deg(u)=\deg(v)=m$, and $m_1+m_2\geq n+1$ if $\deg(u)=m_1$, $\deg(v)=m_2$ where $m_1\neq m_2$. In particular, we must have $uv'-u'v\equiv 0$ if $\deg(uv'-u'v)<\deg(w)$, and in this case u and v are linearly dependent and the curve defined by (3) is therefore a straight line.

Proposition 2. *When the integrals of u^2/w^2 and v^2/w^2 are both rational for $u(\xi),v(\xi),w(\xi)\in\mathbb{R}[\xi]$ satisfying* $\gcd(u,v)=\gcd(u,w)=\gcd(v,w)=1$*, the integral of uv/w^2 is also rational.*

Proof : Recall that, when the integrals of u^2/w^2 and v^2/w^2 are both rational, the condition (8) must hold at each root ξ_k of $w(\xi)$ for $k=1,\ldots,n$. Now $u(\xi_k)\neq 0$ when $\gcd(u,w)=1$, so $w_k u' - w_k' u$ must vanish at ξ_k if u^2/w^2 has a rational integral. Likewise, if $\gcd(v,w)=1$, then $w_k v' - w_k' v$ must vanish at ξ_k when v^2/w^2 has a rational integral. Hence, equation (9) holds at each root ξ_k of $w(\xi)$ for $k=1,\ldots,n$, and this is the condition for the integral of uv/w^2 to be rational. ∎

Example 1. Consider $u(\xi)=\xi^3-2\,\xi^2$, $v(\xi)=\xi^2-\xi+1$, $w(\xi)=\xi(\xi-1)$. Then from (3) with $x(-1)=y(-1)=s(-1)=0$ we obtain

$$x(\xi) \,=\, \frac{\xi^5-4\,\xi^4-3\,\xi^3+7\,\xi^2+2\,\xi-3}{3\,\xi(\xi-1)}\,, \quad y(\xi) \,=\, \frac{3\,\xi^4-9\,\xi^3+12\,\xi}{3\,\xi(\xi-1)}\,, \tag{11}$$

$$s(\xi) \,=\, \frac{\xi^5-4\,\xi^4+3\,\xi^3-2\,\xi^2-7\,\xi+3}{3\,\xi(\xi-1)}\,. \tag{12}$$

Note that w divides $uv'-u'v=-\xi(\xi-1)(\xi^2-\xi+4)$. Since $\gcd(u,v)=1$ the curve has no cusps, as seen in Figure 1. To check that this is an irreducible curve, we write $x(\xi)=X(\xi)/W(\xi)$, $y(\xi)=Y(\xi)/W(\xi)$, and use `Maple` to verify that its implicit equation,

$$f(x,y) \,=\, \mathrm{resultant}_\xi(X(\xi)-x\,W(\xi),Y(\xi)-y\,W(\xi)) \,=\, 0\,,$$

does not factor into lower–order components.

4 Characterization of the residues

Proposition 2 allows us to construct primitive rational PH curves that have rational arc lengths if we can identify, for any given $w(\xi)\in\mathbb{R}[\xi]$ of the form (7) with distinct roots $\xi_1,\ldots,\xi_n\in\mathbb{C}$, pairs of polynomials $u(\xi),v(\xi)\in\mathbb{R}[\xi]$ with $\gcd(u,v)=\gcd(u,w)=\gcd(v,w)=1$, such that u^2/w^2 and v^2/w^2 have rational integrals. Since the same arguments apply to u and v, we focus on the former, and consider polynomials $u(\xi)\in\mathbb{R}[\xi]$ with $\gcd(u,w)=1$ that yield rational functions for the indefinite integral

$$\int \frac{u^2(\xi)}{w^2(\xi)}\,\mathrm{d}\xi\,. \tag{13}$$

The key result is stated in the following Proposition. Since the proof depends on a number of preparatory results, we defer it until later.

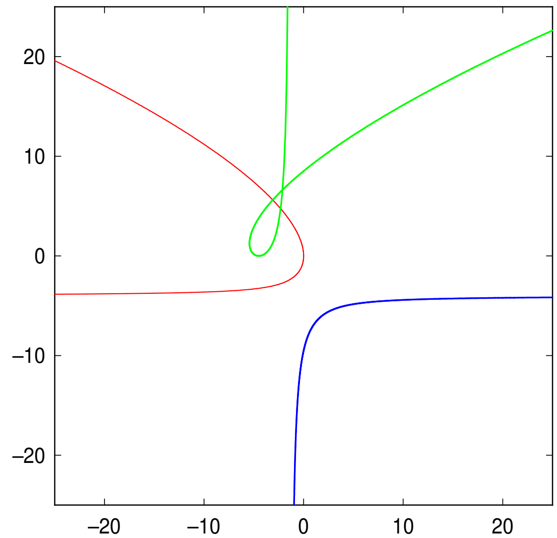

Figure 1: The rational quintic curve with rational arc length in Example 1. The branches of the curve that correspond to the three parameter intervals $\xi \in (-\infty, 0)$, $\xi \in (0, 1)$, and $\xi \in (1, +\infty)$ are plotted in red, blue, and green.

Proposition 3. *For $u(\xi)$ of degree m and $w(\xi)$ of degree n as defined above, the rationality of the integral* (13) *may be categorized as follows.*

Case 1: $m < n$.

- *for even n, the integral* (13) *is rational if and only if $u(\xi) = 0$;*
- *for odd n, there is a unique (modulo a constant factor) polynomial $u(\xi)$ of degree $n - 1$ such that the integral* (13) *is rational.*

Case 2: $m = n$.

- *for even n, the polynomials $u(\xi)$ such that the integral* (13) *is rational differ only by a constant factor;*
- *for odd n, the integral* (13) *is rational if and only if $u(\xi) = 0$.*

Case 3: $m > n$.

- *there are infinitely many polynomials $u(\xi)$ such that the integral* (13) *is rational.*

We begin with a few examples illustrating Propositions 2 and 3.

Example 2. Let $u(\xi) = 3\,\xi^5 - \xi^4 - 7\,\xi^3 + 11\,\xi^2 + 2$, $v(\xi) = 3\,\xi^2 + 1$, and $w(\xi) = \xi(\xi^2 - 1)$. Then we have

$$\frac{u^2}{w^2} = 9\,\xi^4 - 6\,\xi^3 - 23\,\xi^2 + 68\,\xi - 28 + \frac{4}{\xi^2} + \frac{16}{(\xi-1)^2} + \frac{64}{(\xi+1)^2}\,,$$

$$\frac{uv}{w^2} = 9\,\xi - 3 + \frac{2}{\xi^2} + \frac{8}{(\xi-1)^2} + \frac{16}{(\xi+1)^2}\,, \quad \frac{v^2}{w^2} = \frac{1}{\xi^2} + \frac{4}{(\xi-1)^2} + \frac{4}{(\xi+1)^2}\,.$$

Note that $\gcd(u,v) = \gcd(u,w) = \gcd(v,w) = 1$, and the integrals (4) are all rational, in accordance with Proposition 2.

Example 3. Let $u(\xi) = -145\,\xi^4 + 185\,\xi^2 - 292$ and $w(\xi) = \xi(\xi^2-1)(\xi^2-4)$ so that $m < n$ with n odd. Then we have the partial fraction decomposition

$$\frac{u^2}{w^2} = \frac{5329}{\xi^2} + \frac{1764}{(\xi-1)^2} + \frac{1764}{(\xi+1)^2} + \frac{6084}{(\xi-2)^2} + \frac{6084}{(\xi+2)^2}\,,$$

and each term on the right yields a rational expression upon integration.

The following lemma and corollary will be used in proving Proposition 3.

Lemma 1. *For $n \geq 2$, let $\Xi = (\xi_1, \ldots, \xi_n)^T$ be a vector of distinct complex numbers, and let A_n be the $n \times n$ matrix with elements $a_{ii} = 0$ and $a_{ij} = (\xi_i - \xi_j)^{-1}$ for $i \neq j$. Then we have*

(1) rank$(A_n) = n$ when n is even;

(2) rank$(A_n) = n - 1$ when n is odd.

Proof : Observe first that A_n is a skew–symmetric matrix, i.e., $A_n^T = -A_n$.

(1) For even n, we set $n = 2k$ and show by induction on k that $\det(A_n) \neq 0$. When $k = 1$, we have $\det(A_2) = (\xi_1 - \xi_2)^{-2} \neq 0$. For $k > 1$, we set $b_{i-1} = a_{1i}$

for $i = 2, 3, \dots, n$. An easy calculation shows that A_n then has the form

$$\begin{bmatrix} 0 & b_1 & b_2 & b_3 & \cdots & b_{n-2} & b_{n-1} \\ -b_1 & 0 & \dfrac{b_1 b_2}{b_1 - b_2} & \dfrac{b_1 b_3}{b_1 - b_3} & \cdots & \dfrac{b_1 b_{n-2}}{b_1 - b_{n-2}} & \dfrac{b_1 b_{n-1}}{b_1 - b_{n-1}} \\ -b_2 & \dfrac{-b_1 b_2}{b_1 - b_2} & 0 & \dfrac{b_2 b_3}{b_2 - b_3} & \cdots & \dfrac{b_2 b_{n-2}}{b_2 - b_{n-2}} & \dfrac{b_2 b_{n-1}}{b_2 - b_{n-1}} \\ -b_3 & \dfrac{-b_1 b_3}{b_1 - b_3} & \dfrac{-b_2 b_3}{b_2 - b_3} & 0 & \cdots & \dfrac{b_3 b_{n-2}}{b_3 - b_{n-2}} & \dfrac{b_3 b_{n-1}}{b_3 - b_{n-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -b_{n-2} & \dfrac{-b_1 b_{n-2}}{b_1 - b_{n-2}} & \dfrac{-b_2 b_{n-2}}{b_2 - b_{n-2}} & \dfrac{-b_3 b_{n-2}}{b_3 - b_{n-2}} & \cdots & 0 & \dfrac{b_{n-2} b_{n-1}}{b_{n-2} - b_{n-1}} \\ -b_{n-1} & \dfrac{-b_1 b_{n-1}}{b_1 - b_{n-1}} & \dfrac{-b_2 b_{n-1}}{b_2 - b_{n-1}} & \dfrac{-b_3 b_{n-1}}{b_3 - b_{n-1}} & \cdots & \dfrac{-b_{n-2} b_{n-1}}{b_{n-2} - b_{n-1}} & 0 \end{bmatrix}.$$

We now divide rows $2, 3, \dots, n$ and columns $2, 3, \dots, n$ by $b_1, b_2, \dots, b_{n-1}$, respectively, to obtain the matrix A'_n specified by

$$\begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & \dfrac{1}{b_1 - b_2} & \dfrac{1}{b_1 - b_3} & \cdots & \dfrac{1}{b_1 - b_{n-2}} & \dfrac{1}{b_1 - b_{n-1}} \\ -1 & \dfrac{-1}{b_1 - b_2} & 0 & \dfrac{1}{b_2 - b_3} & \cdots & \dfrac{1}{b_2 - b_{n-2}} & \dfrac{1}{b_2 - b_{n-1}} \\ -1 & \dfrac{-1}{b_1 - b_3} & \dfrac{-1}{b_2 - b_3} & 0 & \cdots & \dfrac{1}{b_3 - b_{n-2}} & \dfrac{1}{b_3 - b_{n-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & \dfrac{-1}{b_1 - b_{n-2}} & \dfrac{-1}{b_2 - b_{n-2}} & \dfrac{-1}{b_3 - b_{n-2}} & \cdots & 0 & \dfrac{1}{b_{n-2} - b_{n-1}} \\ -1 & \dfrac{-1}{b_1 - b_{n-1}} & \dfrac{-1}{b_2 - b_{n-1}} & \dfrac{-1}{b_3 - b_{n-1}} & \cdots & \dfrac{-1}{b_{n-2} - b_{n-1}} & 0 \end{bmatrix},$$

and note that

$$\det(A_n) \;=\; \left[\prod_{i=1}^{n-1} b_i^2\right] \det(A'_n)\,.$$

Set $A''_n = A'_n + J_n$, where J_n is the $n \times n$ matrix whose elements are all equal to 1. Then the first column of A''_n is $(1, 0, \dots, 0)^T$ and for even n we have $\det(A''_n) = \det(A'_n)$ — see [10, Problem 1.2, pp. 6 and 25].

Now let B be the $(n-2)\times(n-2)$ matrix with elements[3] $b_{ij} = 1/(b_i - b_j)$ for $i \neq j$ and $b_{ii} = 0$, where $2 \leq i, j \leq n-1$. Then if M denotes the matrix obtained from A''_n by deleting its first two rows and columns, we observe that $M = B + J_{n-2}$, and therefore $\det(M) = \det(B)$ — see [10, Problem 1.2]. By the induction hypothesis, $\det(B) \neq 0$ and this implies that $\text{rank}(B) = n-2$. Hence, the rank of A''_n (and thus of A_n) is at least $n-1$. But the rank of a skew–symmetric matrix is always even [6, Corollary 2.6.6, p. 153] and thus $\text{rank}(A_n) = n$, as required.

(2) For odd n, we have $\det(A_n) = 0$ — see [10, Problem 1.1, pp. 6 and 25]. Let A_{n-1} be the $(n-1)\times(n-1)$ matrix with elements a_{ij} defined in terms of distinct complex numbers $(\xi_1, \ldots, \xi_{n-1})^T$ as above. Then from case (1) we have $\text{rank}(A_{n-1}) = n-1$, since $n-1$ is even. But A_{n-1} is the matrix with an even number of rows and columns obtained from A_n by deleting its last row and column. Thus, $\text{rank}(A_{n-1}) = n-1$, as required. ∎

From Lemma 1, we may immediately deduce the following result.

Corollary 1. *Let the matrices A_n and J_n be as in Lemma 1, and let $M = A_n + c\,J_n$ for $c \in \mathbb{C}$. Then rank$(M) = n$ if $c \neq 0$, and for the system of linear equations $\Sigma : A_n X = C$ with $X = (x_1, x_2, \ldots, x_n)^T$ and $C = (c, c, \ldots, c)^T$ we have:*

(1) Suppose n is odd. If $c \neq 0$, then Σ has no solution, and if $c = 0$, then Σ has for any λ a solution of the form $X = \lambda\,(\rho_1, \rho_2, \ldots, \rho_n)^T$ with $\rho_i \neq 0$ for $i = 1, \ldots, n$.

(2) Suppose n is even. If $c \neq 0$, then Σ has a unique solution of the form $X = (x_1^c, x_2^c, \ldots, x_n^c)^T$ where $x_i^c \neq 0$ for $i = 1, \ldots, n$.

Proof : When n is even, $\text{rank}(A_n) = n$ by Lemma 1, and the determinant of a skew–symmetric matrix of even order remains unchanged when a constant is added to each element [10, Problem 1.2] so $\text{rank}(M) = n$. When n is odd,

[3] Note that $b_i \neq b_j$ for $i \neq j$.

consider the $(n+1)\times(n+1)$ skew–symmetric matrix defined by

$$\Gamma = \begin{bmatrix} 0 & -c & -c & \cdots & -c \\ c & & & & \\ c & & & & \\ \vdots & & & A_n & \\ c & & & & \\ c & & & & \end{bmatrix}.$$

Γ has the same form as the matrix A'_n in the proof of item 1 of Lemma 1, but with the first row and first column multiplied by c, and is thus an invertible matrix. Since $n+1$ is even, we have $\det(\Gamma) = \det(\Gamma + cJ_{n+1})$. However, $\det(\Gamma + cJ_{n+1}) = c\,\det(A_n + cJ_n) \neq 0$, so $\mathrm{rank}(M) = n$.

(1) Suppose Σ has a solution for $c \neq 0$. Then the rank of the matrix A_n augmented by the column vector C is equal to the rank of A_n, namely $n-1$. Now if we subtract C from each column of A_n, we obtain the matrix $A_n - c\,J_n$, whose rank must be $n-1$. But the latter matrix has rank n from item (1), a contradiction.

Now suppose $c = 0$. Since $\mathrm{rank}(A_n) = n-1$, Σ has a one–dimensional solution of the form $X = \lambda\,(\rho_1, \rho_2, \ldots, \rho_n)^T$. Finally, if $\rho_i = 0$ for some i, we may assume (by rearranging the order of the ξ_i) that $i = 1$. Then, if B is the $(n-1)\times(n-1)$ matrix defined in terms of the vector $(b_2, b_3, \ldots, b_{n-1})^T$ as in the proof of Lemma 1, $X_1 = \lambda\,(\rho_2, \ldots, \rho_n)^T$ is a solution of the system $B\,X = (0, \ldots, 0)^T$. But since $\mathrm{rank}(B) = n-1$, this system has only the trivial solution, a contradiction.

(2) Since $\mathrm{rank}(A_n) = n$, the matrix A_n is invertible, and the system Σ has a unique solution $X = (x_1^c, x_2^c, \ldots, x_n^c)^T = A_n^{-1}C$ for each $c \neq 0$, with $x_i^c \neq 0$ by an argument similar to that used in item (1). ∎

The motivation for Lemma 1 is as follows. For $w(\xi)$ of the form (7) and a polynomial $u(\xi) \in \mathbb{R}[\xi]$ of degree m, the partial fraction decomposition of $u(\xi)/w(\xi)$ is

$$\frac{u(\xi)}{w(\xi)} = p(\xi) + \sum_{i=1}^{n} \frac{r_i}{(\xi - \xi_i)}\,, \quad \text{with } r_i = \frac{u(\xi_i)}{w'(\xi_i)}\,, \tag{14}$$

where $p(\xi)$ is 0 if $m < n$; a constant c if $m = n$; and a polynomial of degree $m - n$ if $m > n$. Note that $\gcd(u, w) = 1$ if and only if the residues satisfy $r_i \neq 0$ for $i = 1, \ldots, n$. Now from (14) we obtain

$$\frac{u^2(\xi)}{w^2(\xi)} \;=\; p^2(\xi) + \sum_{i=1}^{n} \frac{r_i^2}{(\xi - \xi_i)^2} + \sum_{i<j} \frac{2\, r_i r_j}{(\xi - \xi_i)(\xi - \xi_j)} + \sum_{i=1}^{n} \frac{2\, r_i\, p(\xi)}{(\xi - \xi_i)}\,. \tag{15}$$

Using (5), we can now compute the residues of (15) at $\xi = \xi_k$ for $k = 1, \ldots, n$. The residues of the first two terms in (15) at these poles are zero. To evaluate the residues of the third term, we note that

$$\frac{1}{(\xi - \xi_i)(\xi - \xi_j)} \;=\; \frac{a_{ij}}{\xi - \xi_i} + \frac{a_{ji}}{\xi - \xi_j}\,,$$

where $a_{ij} = 1/(\xi_i - \xi_j)$, and the residue of this expression at $\xi = \xi_k$ is just $a_{ij}\delta_{ik} + a_{ji}\delta_{jk}$, where the Kronecker symbol δ_{rs} is equal to 1 if $r = s$ and 0 otherwise. From this, the residue of the third term can be expressed as

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2\, r_i r_j (a_{ij}\delta_{ik} + a_{ji}\delta_{jk})\,.$$

Since only the indices $i = k < j$ and $i < k = j$ make non–zero contributions to this expression, it may be reduced[4] to

$$2\, r_k \sum_{j=k+1}^{n} a_{kj} r_j \;+\; 2\, r_k \sum_{i=1}^{k-1} a_{ki} r_i\,.$$

Combining these sums yields the residue of the third term at $\xi = \xi_k$ as

$$2\, r_k \sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} r_j\,.$$

Finally, the residue of the fourth term in (15) is $2\, p(\xi_k) r_k$. Thus, combining the residues of the third and fourth terms, and cancelling the common factor $2\, r_k$ (since $r_k \neq 0$ for $k = 1, \ldots, m$), the condition for (15) to have a rational integral reduces to the system of linear equations

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} r_j \;=\; -\, p(\xi_k)\,, \quad k = 1, \ldots, n$$

[4]Note that, with $1 \leq i < j \leq n$, only one of the terms in $a_{ij}\delta_{ik} + a_{ji}\delta_{jk}$ cannot vanish.

for the residues $r_1, \ldots, r_n$ of $u(\xi)/w(\xi)$. Upon setting $R = (r_1, \ldots, r_n)^T$ and $P = -\,(p(\xi_1), \ldots, p(\xi_n))^T$, this can be expressed in matrix form as

$$A_n R \,=\, P\,, \tag{16}$$

where the skew–symmetric matrix A_n is defined in Lemma 1.

For a given P, a solution $R = (r_1, \ldots, r_n)^T$ of the system (16) is called a *residues vector*, and is said to be a *good solution* when $r_i \neq 0$ for $i = 1, \ldots, n$ and the expression (14) generates a polynomial $u(\xi) \in \mathbb{R}[\xi]$. Note that the polynomials $u(\xi)$ and $v(\xi)$ constructed from two linearly–independent good solutions must be verified to satisfy the condition $\gcd(u, v) = 1$, as required in Proposition 2 for all of the integrals (4) to be rational.

Remark 3. Note that the solutions to (16) are independent of the *ordering* of the roots of $w(\xi)$. For any given root vector $\Xi = (\xi_1, \ldots, \xi_n)^T$ let $\tilde{\Xi} = \Pi\,\Xi$ be another ordering, specified by a permutation matrix Π with $\Pi^T = \Pi^{-1}$. Then the matrices A_n and $\tilde{A}_n$ associated with the root vectors Ξ and $\tilde{\Xi}$ are related by $\tilde{A}_n = \Pi\, A_n\, \Pi^T$. Consequently, if R is a solution of (16) for a given P, then $\tilde{R} = \Pi\, R$ is a solution of $\tilde{A}_n \tilde{R} = \tilde{P}$ where $\tilde{P} = \Pi\, P$.

We show that for certain P the system (16) always admits good solutions. If the roots of $w(\xi)$ are all real, any vector $R = (r_1, \ldots, r_n)^T$ with real $r_i \neq 0$ for $i = 1, \ldots, n$ is a good solution, since a real $u(\xi)$ can be constructed from (14) with $p(\xi)$ matching values $p(\xi_1), \ldots, p(\xi_n)$ obtained from $P = A_n R$. If $w(\xi)$ has complex roots,[5] we order them as $\Xi = (\xi_1, \ldots, \xi_k, \boldsymbol{\xi}_1, \boldsymbol{\xi}_1^*, \cdots, \boldsymbol{\xi}_l, \boldsymbol{\xi}_l^*)^T$ where $k + 2\,l = n$, such that $\xi_i \in \mathbb{R}$ for $i = 1, \ldots, k$ and $\boldsymbol{\xi}_i, \boldsymbol{\xi}_i^* \in \mathbb{C} \setminus \mathbb{R}$ are conjugate pairs for $i = 1, \ldots, l$, and let A_n be the matrix constructed from them as in Lemma 1. Then the following result provides a sufficient condition for a good solution.

Lemma 2. *For* $\Xi = (\xi_1, \ldots, \xi_k, \boldsymbol{\xi}_1, \boldsymbol{\xi}_1^*, \cdots, \boldsymbol{\xi}_l, \boldsymbol{\xi}_l^*)^T$ *we have:*

1. *Any vector of the form* $R = (r_1, \ldots, r_k, \mathbf{r}_1, \mathbf{r}_1^*, \ldots, \mathbf{r}_l, \mathbf{r}_l^*)^T$ *with* $r_i \neq 0 \in \mathbb{R}$ *for* $i = 1, \ldots, k$ *and* $\mathbf{r}_j \neq 0 \in \mathbb{C}$ *for* $i = 1, \ldots, l$ *is a good solution.*

2. *Let* $P = (p_1, \ldots, p_k, \mathbf{p}_1, \mathbf{p}_1^*, \ldots, \mathbf{p}_l, \mathbf{p}_l^*)^T$ *with* $p_i \in \mathbb{R}$ *for* $i = 1, \ldots, k$ *and* $\mathbf{p}_i, \mathbf{p}_i^* \in \mathbb{C}$ *for* $i = 1, \ldots, l$ *if* n *is even, and* $P = (0, \ldots, 0)^T$ *if* n *is odd. Then if the system* (16) *has a solution* $R = (r_1, \ldots, r_n)^T$ *with* $r_i \neq 0$ *for*

[5] For clarity, italic and bold characters are employed here to distinguish between real and non–real values.

$i = 1, \ldots, n$, it must be of the form $(r_1, \ldots, r_k, \mathbf{r}_1, \mathbf{r}_1^*, \ldots, \mathbf{r}_l, \mathbf{r}_l^*)^T$ where $r_i \in \mathbb{R}$ for $i = 1, \ldots, k$ and $\mathbf{r}_i, \mathbf{r}_i^* \in \mathbb{C}$ for $i = 1, \ldots, l$.

Proof. Let $\tilde{\Xi} = (\xi_1, \ldots, \xi_k, \boldsymbol{\xi}_1^*, \boldsymbol{\xi}_1, \ldots \boldsymbol{\xi}_l^*, \boldsymbol{\xi}_l)^T$ be a re–ordering of the roots of $w(\xi)$ and let $\tilde{A}_n$ be the matrix constructed from $\tilde{\Xi}$ as in Lemma 1. Note that $\tilde{\Xi} = \Xi^*$ and $\tilde{A}_n = A_n^*$. Also, let $\tilde{R} = (r_1, \ldots, r_k, \mathbf{r}_1^*, \mathbf{r}_1, \ldots, \mathbf{r}_l^*, \mathbf{r}_l)^T = R^*$.

1. If $A_n R = P = (p_1, \ldots, p_k, \mathbf{x}_1, \mathbf{y}_1, \ldots, \mathbf{x}_l, \mathbf{y}_l)^T$, then $\tilde{A}_n \tilde{R} = (p_1, \ldots, p_k, \mathbf{y}_1, \mathbf{x}_1, \ldots, \mathbf{y}_l, \mathbf{x}_l)^T$. But $A_n^* R^* = P^* = (p_1, \ldots, p_k, \mathbf{x}_1^*, \mathbf{y}_1^*, \ldots, \mathbf{x}_l^*, \mathbf{y}_l^*)^T$, and since $\tilde{A}_n = A_n^*$ and $\tilde{R} = R^*$, the values $\mathbf{x}_i, \mathbf{y}_i$ must be conjugates, so P is of the form $(p_1, \ldots, p_k, \mathbf{p}_1, \mathbf{p}_1^*, \ldots, \mathbf{p}_l, \mathbf{p}_l^*)^T$. Therefore, $p(\xi) \in \mathbb{R}[\xi]$ since $p(\xi_i) = -p_i$ for $i = 1, \ldots, k$ and $p(\boldsymbol{\xi}_i) = -\mathbf{p}_i$, $p(\boldsymbol{\xi}_i^*) = -\mathbf{p}_i^*$ for $i = 1, \ldots, l$. Also, the sum on the right in (14) is real, since the complex terms occur in conjugate pairs. Hence, $u(\xi)$ is a real polynomial.

2. Let $R = (r_1, \ldots, r_k, \mathbf{a}_1, \mathbf{b}_1, \ldots, \mathbf{a}_l, \mathbf{b}_l)^T$ be a solution of (16) with non–zero elements. Then with $\tilde{R} = (r_1, \ldots, r_k, \mathbf{b}_1, \mathbf{a}_1, \ldots, \mathbf{b}_l, \mathbf{a}_l)^T$ and $P^* = (p_1, \ldots, p_k, \mathbf{p}_1^*, \mathbf{p}_1, \ldots, \mathbf{p}_l^*, \mathbf{p}_l)^T$ we have $\tilde{A}_n \tilde{R} = P^*$. Since $\tilde{A}_n = A^*$, this implies that $A_n^* \tilde{R} = P^*$ and consequently $A_n \tilde{R}^* = P$. Hence, we obtain $A_n R = A_n \tilde{R}^* = P$. If n is even, A_n is invertible, and we have $\tilde{R}^* = R$, so R has the stated form. If n is odd, $A_n R = P$ has a one–dimensional solution of the form $R = \lambda(\rho_1, \ldots, \rho_n)^T$ by item 2 of Corollary 1. Hence, $R = \lambda_1(\rho_1, \ldots, \rho_n)^T$ and $\tilde{R}^* = \lambda_2(\rho_1, \ldots, \rho_n)^T$. Since n is odd $w(\xi)$ must have at least one real root, which implies that $r_1 \in \mathbb{R}$, and consequently $\tilde{R}^* = R$, so R has the stated form.

We are now ready to proceed with the proof of Proposition 3.

Proof :

Case 1: $m < n$. In this case, note that $P = (0, \ldots, 0)^T$ since $p(\xi) \equiv 0$.

- If n is even, then A_n is invertible, and from (16) we have $R = (0, \ldots, 0)^T$. Consequently, from (14) we deduce that $u(\xi) = 0$.
- If n is odd, then by item 2 of Corollary 1 and item 2 of Lemma 2, equation (16) has the good solution $R = \lambda\,(\rho_1, \rho_2, \ldots, \rho_n)^T$ with $\lambda, \rho_i \in \mathbb{C}$, so $u(\xi)$ is unique up to a constant factor as required.

Case 2: $m = n$. In this case, we take $p(\xi) = c \in \mathbb{R}$, with $c \neq 0$ for $u(\xi) \neq 0$.

- If n is even, then $R = A_n^{-1} C$ with $C = (c, c, \ldots, c)^T$ is a good solution, as indicated by item 3 of Corollary 1 and item 2 of Lemma 2. Hence, the polynomials $u(\xi)$ differ only by a constant factor.

- If n is odd, by item 2 of Corollary 1, equation (16) has no solution when $c \neq 0$. A solution exists only when $c = 0$, corresponding to $u(\xi) = 0$.

Case 3: $m > n$. If the roots $\xi_1, \ldots, \xi_n$ of $w(\xi)$ are all real, then any vector $R = (r_1, \ldots, r_n)^T \in \mathbb{R}^n$ with $r_i \neq 0$, $i = 1, \ldots, n$ is a good solution. If $w(\xi)$ has complex conjugate roots, any vector $R = (r_1, \ldots, r_k, \mathbf{r}_1, \mathbf{r}_1^*, \ldots, \mathbf{r}_l, \mathbf{r}_l^*)^T$ of the form in Lemma 2, with non–zero elements, is a good solution. Hence, there are infinitely many polynomials $u(\xi)$ such that (13) is rational. ■

Case 3 of Proposition 3 is of primary interest, since we require linearly–independent polynomials $u(\xi), v(\xi)$ to define a non–trivial rational PH curve $\mathbf{r}(\xi)$ with rational arc length. Note that, from any two linearly–independent polynomials $u(\xi), v(\xi)$ such that the integrals (4) are rational for a prescribed $w(\xi)$, we may construct further examples by means of Remark 1.

We focus here on the construction of $u(\xi)$, the construction of $v(\xi)$ being exactly analogous. Defining a monic polynomial $w(\xi)$ of degree n by distinct real roots $\xi_1, \ldots, \xi_n$, we construct the matrix A_n in Lemma 1, and for $m > n$ we assign values $R = (r_1, \ldots, r_n)^T$ for the residues that define a good solution of the linear system (16). The polynomial $p(\xi)$ is then constructed from the values $P = (-p(\xi_1), \ldots, -p(\xi_n))^T = A_n R$ obtained from (16) as a Lagrange interpolant. Finally, we recover $u(\xi)$ using (14) as

$$u(\xi) \,=\, w(\xi) \left[p(\xi) + \sum_{i=1}^{n} \frac{r_i}{(\xi - \xi_i)} \right] . \tag{17}$$

Note that, by Lemma 2, $u(\xi)$ is a *real* polynomial if complex conjugate values are assigned to the residues, since equation (16) furnishes complex conjugate values for $p(\xi)$ at the corresponding complex conjugate roots of $w(\xi)$.

A second polynomial $v(\xi)$ may be constructed in the same manner, and it must be verified that $\gcd(u, v) = 1$ to satisfy the conditions of Proposition 2, and ensure that $u(\xi)$ and $v(\xi)$ are linearly independent. The curve $\mathbf{r}(\xi)$ and its arc length function $s(\xi)$ are then constructed by substituting $u(\xi), v(\xi), w(\xi)$ into expressions (3). The following examples serve to illustrate the procedure in more concrete terms.

Example 4. Consider a case with n odd, namely $w(\xi) = \xi(\xi^2 - 1)(\xi^2 - 4)$ with roots $\Xi = (0, 1, -1, 2, -2)^T$. Then we obtain the matrix

$$A_5 = \begin{bmatrix} 0 & -1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & -1 & \frac{1}{3} \\ -1 & -\frac{1}{2} & 0 & -\frac{1}{3} & 1 \\ \frac{1}{2} & 1 & \frac{1}{3} & 0 & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{3} & -1 & -\frac{1}{4} & 0 \end{bmatrix},$$

and $\operatorname{rank}(A_5) = 4$ by Lemma 1. Hence, the system (16) has a solution if and only if the matrix A_5 augmented by the column vector $P = (p_1, p_2, p_3, p_4, p_5)^T$ is also of rank 4, where $p_i = -p(\xi_i)$ for $i = 1, \dots, 5$. By Gaussian elimination to reduce this augmented matrix to row echelon form, the condition for it to be of rank 4 may be identified as

$$73\,p_1 - 42\,p_2 - 42\,p_3 + 78\,p_4 + 78\,p_5 = 0\,. \tag{18}$$

For the choice $P = (0, 0, 0, 0, 0)^T$ the linear system (16) has the good solution $R = \lambda\,(-73, 42, 42, -78, 78)^T$ where $\lambda \in \mathbb{R}$. For this case, $p(\xi) \equiv 0$ and from (17) we obtain $u(\xi) = -145\,\xi^4 + 185\,\xi^2 - 292$. Then from Proposition 3, this is (modulo a constant factor) the unique polynomial of degree ≤ 4 for which $\gcd(u, w) = 1$ and the integral (13) is rational.

For $P = (0, 2, -2, 1, -1)^T/12$ satisfying (18), the system (16) has the good solution $R = (-437, 252, 252, -468, -468)^T/6$. We have $p(\xi) = (\xi^3 - 5\,\xi)/24$ such that $p(\xi_i) = -p_i$ for $i = 1, \dots, 5$ and we construct

$$v(\xi) = w(\xi)\left[p(\xi) + \sum_{i=1}^{5} \frac{R_i}{(\xi - \xi_i)}\right] = \frac{\xi^8 - 10\,\xi^6 - 3447\,\xi^4 + 4400\,\xi^2 - 6992}{24}\,.$$

Note that $\gcd(v, w) = 1$, and the integral of $v^2(\xi)/w^2(\xi)$ is rational. We also have $\gcd(u, v) = 1$, with $u(\xi)$ as defined above.

To illustrate that further solutions of the system (16) are possible, we consider the values $P = (6, 1, 2, -2, -2)^T/12$ satisfying (18). Then we obtain the good solution $R = (881, -85, -504, 942, 942)^T/12$. Using the same procedure as above, we have $p(\xi) = (-5\,\xi^4 - \xi^3 + 32\,\xi^2 + 4\,\xi - 36)/72$ and we construct the polynomial $h(\xi)$ given by

$$\frac{-5\,\xi^9 - \xi^8 + 57\,\xi^7 + 9\,\xi^6 - 216\,\xi^5 + 10482\,\xi^4 + 272\,\xi^3 - 13382\,\xi^2 + 21144}{72}\,.$$

Then the integral of $h^2(\xi)/w^2(\xi)$ is rational, and we observe that $\gcd(h,w) = \gcd(h,u) = \gcd(h,v) = 1$.

Example 5. Consider a case with n even, namely $w(\xi) = \xi(\xi^2-1)(\xi-2)$ with roots $\Xi = (0,-1,1,2)^T$. Then the matrix

$$A_4 = \begin{bmatrix} 0 & 1 & -1 & -\frac{1}{2} \\ -1 & 0 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & \frac{1}{2} & 0 & -1 \\ \frac{1}{2} & \frac{1}{3} & 1 & 0 \end{bmatrix}$$

has rank 4 by Lemma 1, so the system (16) has a solution for any $P \in \mathbb{R}^4$.

For the values $P = (1,1,1,1)^T$ the linear system (16) has the good solution $R = (-14,18,14,-18)^T/13$. In this case $p(\xi) \equiv -1$, and from (14) we obtain

$$u(\xi) = \frac{-13\,\xi^4 + 26\,\xi^3 - 27\,\xi^2 + 14\,\xi - 28}{13}.$$

By Proposition 3, this is the unique polynomial (modulo a constant factor) of degree ≤ 4 for which $\gcd(u,w) = 1$ and the integral (13) is rational.

For the residues vector $R = (-1,3,2,3)^T$ we obtain the corresponding values $P = (-1,-2,-5,5)^T/2$, and the interpolation conditions $p(\xi_i) = -p_i$ for $i = 1,\ldots,4$ yield

$$p(\xi) = \frac{-19\,\xi^3 + 15\,\xi^2 + 28\,\xi + 6}{12}.$$

Then we construct the polynomial

$$v(\xi) = \frac{-19\,\xi^7 + 53\,\xi^6 + 17\,\xi^5 - 103\,\xi^4 + 74\,\xi^3 - 58\,\xi^2 + 12\,\xi - 24}{12}$$

such that $\gcd(v,w) = 1$ and the integral of $v^2(\xi)/w^2(\xi)$ is rational. Also note that $\gcd(u,v) = 1$, where $u(\xi)$ is as defined above.

Example 6. Consider the polynomial $w(\xi) = \xi^2 + 1$ with complex conjugate roots $\Xi = (\mathrm{i}, -\mathrm{i})^T$, for which we have the matrix

$$A_2 = \begin{bmatrix} 0 & -\frac{1}{2}\,\mathrm{i} \\ \frac{1}{2}\,\mathrm{i} & 0 \end{bmatrix}.$$

Choosing $P = (1+\mathrm{i}, 1-\mathrm{i})^T$ yields the linear polynomial $p(\xi) = \xi + 1$, and solving (16) gives $R = (2+2\,\mathrm{i}, 2-2\,\mathrm{i})^T$ for the residues. Then, invoking (17), we construct the polynomial $u(\xi) = \xi^3 + \xi^2 + 5\,\xi - 3$. On the other hand, by choosing $P = (2-\mathrm{i}, 2+\mathrm{i})^T$ we obtain $p(\xi) = -\,\xi + 2$, and with the residues $R = (-\,2+4\,\mathrm{i}, -\,2-4\,\mathrm{i})^T$ we obtain the polynomial $v(\xi) = -\,\xi^3 + 2\,\xi^2 - 5\,\xi - 6$ from (17). We observe that $\gcd(u, w) = \gcd(v, w) = \gcd(u, v) = 1$, and by substituting into (3) and choosing $x(0) = y(0) = s(0) = 0$, we obtain the rational quintic curve defined by

$$\begin{aligned} W(\xi) &= 3\,(\xi^2 + 1)\,, \\ X(\xi) &= 9\,\xi\,(\xi^3 - \xi^2 - 15\,\xi - 9)\,, \\ Y(\xi) &= \xi\,(-\,2\,\xi^4 + 3\,\xi^3 - 38\,\xi^2 - 45\,\xi + 108)\,, \end{aligned}$$

which has the rational arc length function

$$s(\xi) = \frac{\xi\,(2\,\xi^4 - 3\,\xi^3 + 65\,\xi^2 + 45\,\xi + 135)}{3\,(\xi^2 + 1)}\,.$$

The constructed curve is shown in Figure 2. It may be regarded as the sum of the cubic polynomial curve $(x(\xi), y(\xi)) = (3\,\xi^2 - 3\,\xi - 48, -\frac{2}{3}\xi^3 + \xi^2 - 12\,\xi - 16)$ and the quadratic rational curve $(x(\xi), y(\xi)) = (-24\,\xi + 48, 48\,\xi + 16)/(\xi^2 + 1)$.

Example 7. Consider the polynomial $w(\xi) = \xi(\xi - 1)(\xi^2 + 1)$ with the roots $\Xi = (0, 1, \mathrm{i}, -\mathrm{i})^T$ for which we obtain the matrix

$$A_4 = \begin{bmatrix} 0 & -1 & \mathrm{i} & -\mathrm{i} \\ 1 & 0 & \frac{1}{2}(1+\mathrm{i}) & \frac{1}{2}(1-\mathrm{i}) \\ -\mathrm{i} & -\frac{1}{2}(1+\mathrm{i}) & 0 & -\frac{1}{2}\,\mathrm{i} \\ \mathrm{i} & -\frac{1}{2}(1-\mathrm{i}) & \frac{1}{2}\,\mathrm{i} & 0 \end{bmatrix}.$$

In accordance with Lemma 2, we take $P = (1, 1, 1+\mathrm{i}, 1-\mathrm{i})^T$ and this yields the polynomial $p(\xi) = -\,\frac{1}{2}\,\xi^3 + \frac{1}{2}\,\xi + 1$. Solving (16), we obtain the solution $R = (2, 3, -3-\mathrm{i}, -3+\mathrm{i})^T$ for the residues. Finally, using (17), we construct the polynomial

$$u(\xi) = \frac{-\,\xi^7 + \xi^6 + 2\,\xi^4 - 5\,\xi^3 + 15\,\xi^2 + 2\,\xi - 2}{2}\,.$$

One can verify that $\gcd(u, w) = 1$ and $u^2(\xi)/w^2(\xi)$ has a rational integral. By a different choice of P, consistent with Lemma 2, we may obtain another polynomial $v(\xi)$ with $\gcd(v, w) = \gcd(u, v) = 1$, and thus construct a rational curve with rational arc length by substituting into (3).

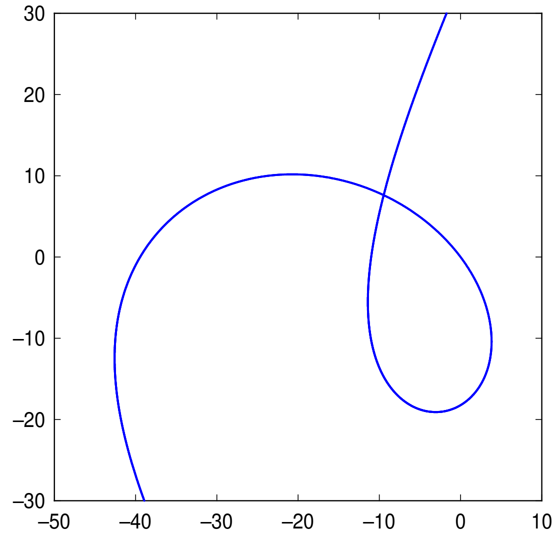

Figure 2: The rational quintic curve with rational arc length in Example 6.

5 Evolutes of planar rational PH curves

An explicit characterization of planar rational PH curves $\mathbf{r}(\xi) = (x(\xi), y(\xi))$ that admit rational arc length functions $s(\xi)$ was identified by Pottmann [9]. Namely $x(\xi), y(\xi), s(\xi)$ are given in terms of two relatively prime polynomials $a(\xi), b(\xi)$ and a rational function $h(\xi)$ by the expressions

$$x \,=\, \frac{b^2 - a^2}{a^2 + b^2}\, H \,-\, \frac{ab}{a'b - ab'}\, H' \,, \quad y \,=\, \frac{2\,ab}{a^2 + b^2}\, H \,-\, \frac{a^2 - b^2}{2\,(a'b - ab')}\, H' \,, \quad (19)$$

$$s \,=\, h \,+\, \frac{a^2 + b^2}{2\,(a'b - ab')}\, H' \,+\, \text{constant}\,, \quad (20)$$

where $H(\xi)$ is the rational function defined by

$$H \,=\, \frac{a^2 + b^2}{2\,(a'b - ab')}\, h' \,. \quad (21)$$

These curves are identified in [9] as the *evolutes* (loci of centers of curvature) of general planar rational PH curves. For example, the rational PH quartics are involutes of the unique polynomial PH cubic, the *Tschirnhaus cubic* [4].

Setting $h = f/g$ for polynomials with $\gcd(f,g) = 1$, we have

$$H \,=\, \frac{(a^2+b^2)(f'g-fg')}{2\,(a'b-ab')\,g^2} \quad \text{and} \quad H' \,=\, \frac{q}{2\,(a'b-ab')^2 g^3}\,,$$

where the polynomial q can be written as

$$\begin{aligned} q \,=\, & (a^2+b^2)\,[\,(a'b-ab')(f''g-fg'')-(a''b-ab'')(f'g-fg')\,]\,g \\ & +\, 2\,(a'b-ab')(f'g-fg')\,[\,(aa'+bb')\,g-(a^2+b^2)\,g'\,]\,. \end{aligned} \tag{22}$$

On substituting into (19), the homogeneous coordinates of the curve $(x,y) = (X/W, Y/W)$ can be expressed as

$$\begin{aligned} W \,&=\, 4\,[\,(a'b-ab')\,g\,]^3\,, \\ X \,&=\, 2\,(b^2-a^2)(a'b-ab')^2(f'g-fg')\,g - 2\,ab\,q\,, \\ Y \,&=\, 4\,ab\,(a'b-ab')^2(f'g-fg')\,g-(a^2-b^2)\,q\,. \end{aligned} \tag{23}$$

Hence, the points at infinity correspond to the roots of $(a'b-ab')\,g$, and they appear to be (at least) triple points. The corresponding arc length (20) is

$$s \,=\, \frac{2\,(a'b-ab')\,fg+(a^2+b^2)(f'g-fg')}{2\,(a'b-ab')\,g^2} \,+\, \text{constant}\,. \tag{24}$$

Note that, whereas expressions (19)–(20) involve *four* polynomials (a, b and the numerator and denominator of h) the form (3) depends on only the *three* polynomials u, v, w. However, the latter requires u, v to be chosen *a priori* for a given w so as to ensure rationality of the integrals (4).

If any of a, b or g are of degree ≥ 2, then w has (in general) more than one distinct root, so the expressions (23) define a proper rational curve with more than one point at infinity (i.e., not a re–parameterized polynomial curve).

The form (23) of a plane rational curve $\mathbf{r}(\xi) = (x(\xi), y(\xi))$ with a rational arc length is constructed in [9] by considering the family of osculating planes to the (spatial) *curve of constant slope* $\mathbf{s}(\xi) = (x(\xi), y(\xi), s(\xi))$, wherein the rational function $s(\xi)$ is specified as the arc length of the curve $\mathbf{r}(\xi)$. However, as the following example illustrates, it is not difficult to construct instances of rational curves with rational arc lengths, through direct integration of (2), that (without simplification) are incompatible with the form (23).

The focus of this study has been on cases in which $w(\xi)$ in (3) has only simple roots, since the analysis is more involved if $w(\xi)$ has multiple roots. We provide the following example to illustrate the existence of rational curves with rational arc lengths when the roots of $w(\xi)$ are not all simple.

Example 8. Consider the rational PH curve $\mathbf{r}(\xi) = (x(\xi), y(\xi))$ constructed by integrating the expressions

$$x'(\xi) \,=\, \frac{u^2(\xi) - v^2(\xi)}{w^2(\xi)}\,, \quad y'(\xi) \,=\, \frac{2\,u(\xi)v(\xi)}{w^2(\xi)}\,, \tag{25}$$

with $u(\xi) = \xi^2 - 4\,\xi + 2$, $v(\xi) = -\,2\,\xi + 1$, $w(\xi) = \xi^2(\xi - 1)$. Note that, with these choices, the integrals (4) are rational, although $w(\xi)$ has a double root. Now since $\xi = 0$ and $\xi = 1$ identify points at infinity on $\mathbf{r}(\xi)$, we take integration constants such that $x(-1) = y(-1) = 0$, and the homogeneous coordinate polynomials

$$\begin{aligned} W(\xi) \,&=\, 3\,\xi^3(\xi - 1)\,, \\ X(\xi) \,&=\, -\,15\,\xi^4 + 12\,\xi^3 + 12\,\xi^2 - 12\,\xi + 3\,, \\ Y(\xi) \,&=\, -\,13\,\xi^4 + 13\,\xi^3 + 6\,\xi^2 - 16\,\xi + 4\,, \end{aligned} \tag{26}$$

define the resulting rational PH curve $\mathbf{r}(\xi) = (X(\xi)/W(\xi), Y(\xi)/W(\xi))$. The parametric speed $\sigma(\xi) = |\mathbf{r}'(\xi)|$, satisfying $x'^2(\xi) + y'^2(\xi) = \sigma^2(\xi)$, is specified by the rational function

$$\sigma(\xi) \,=\, \frac{\xi^4 - 8\,\xi^3 + 24\,\xi^2 - 20\,\xi + 5}{\xi^4(\xi - 1)^2}\,.$$

Since the numerator has only complex conjugate roots, $\sigma(\xi) > 0$ for all ξ and the curve is free of cusps. The curve arc length $s(\xi)$, obtained by integrating $\sigma(\xi)$ with $s(-1) = 0$, is

$$s(\xi) \,=\, \frac{-\,20\,\xi^4 + 17\,\xi^3 + 12\,\xi^2 - 20\,\xi + 5}{3\,\xi^3(\xi - 1)}\,. \tag{27}$$

Using the method described in Example 1, it was verified that the expressions (26) define an irreducible algebraic curve.

At first sight, the curve (26) seems inconsistent with the formulation (23), since $W(\xi)$ is evidently not the perfect cube of a polynomial. The resolution of this paradox requires a detailed analysis, through which the appropriate polynomials $a(\xi), b(\xi)$ and $f(\xi), g(\xi)$ to be used in (23) are determined.

The rational curves with rational arc lengths were identified in [9] as the evolutes of general rational PH curves. Conversely, the general rational PH curves are involutes of rational curves with rational arc lengths. The involute

$\mathbf{r}_i(\xi)$ corresponding to parameter value $\xi = \tau$ of a curve $\mathbf{r}(\xi)$ with arc length function $s(\xi)$ and unit tangent $\mathbf{t}(\xi)$ can be expressed as

$$\mathbf{r}_i(\xi) \,=\, \mathbf{r}(\xi) - [\, s(\xi) - s(\tau)\,]\,\mathbf{t}(\xi)\,. \tag{28}$$

It may be viewed as the locus traced by the end of a string wrapped around $\mathbf{r}(\xi)$ for $\xi \geq \tau$, that is kept taut as it is unwrapped from $\mathbf{r}(\xi)$. The involute of the curve $\mathbf{r}(\xi) = (x(\xi), y(\xi))$ specified by the homogeneous coordinates (26), with unit tangent $\mathbf{t}(\xi) = (x'(\xi), y'(\xi))/\sigma(\xi)$ and the arc length function (27) was computed symbolically using the `Maple` computer algebra system.

Since the resulting symbolic expressions are rather cumbersome, we focus on the representative case $\tau = -1$, for which the involute $\mathbf{r}_i(\xi)$ specified by (28) is a rational quintic, defined by the homogeneous coordinates

$$\begin{aligned}
W_i(\xi) &= 3\,\xi\,(\xi^{\,4} - 8\,\xi^3 + 24\,\xi^2 - 20\,\xi + 5)\,, \\
X_i(\xi) &= (\xi+1)^2(5\,\xi^3 - 50\,\xi^2 + 55\,\xi - 16)\,, \\
Y_i(\xi) &= (\xi+1)^2(-\,13\,\xi^3 + 50\,\xi^2 - 45\,\xi + 12)\,,
\end{aligned} \tag{29}$$

and it is a rational PH curve, since it has the rational parametric speed

$$\sigma_i(\xi) \,=\, \frac{4\,(\xi+1)\,(20\,\xi^3 - 37\,\xi^2 + 25\,\xi - 5)}{3\,\xi^2(\xi^4 - 8\,\xi^3 + 24\,\xi^2 - 20\,\xi + 5)}\,. \tag{30}$$

We will show that the evolute of the curve (29), defined by the homogeneous coordinates (26), is consistent with the expressions (23) through cancellation of a factor common to W, X, Y. In order to do this, we use `Maple` to identify the appropriate polynomials $a(\xi), b(\xi)$ and $f(\xi), g(\xi)$ in (23).

The tangent and normal to a rational PH curve are specified in [9] as

$$\mathbf{t} \,=\, (t_x, t_y) \,=\, \frac{(a^2 - b^2, -\,2\,ab)}{a^2 + b^2}\,, \quad \mathbf{n} \,=\, (n_x, n_y) \,=\, \frac{(2\,ab, a^2 - b^2)}{a^2 + b^2}\,.$$

Equating $\mathbf{t}$ to the expression obtained by substituting (29) and (30) into

$$\mathbf{t} \,=\, \frac{(W_iX_i' - W_i'X_i, W_iY_i' - W_i'Y_i)}{\sigma_i W_i^2}\,,$$

we obtain a and b as

$$a(\xi) \,=\, \frac{\xi^2 - 2\,\xi + 1}{\sqrt{2}}\,, \quad b(\xi) \,=\, \frac{-\,\xi^2 + 6\,\xi - 3}{\sqrt{2}}\,. \tag{31}$$

Now the equation of the tangent line at the point ξ of the involute is

$$n_x(\xi)\left[x - \frac{X_i(\xi)}{W_i(\xi)}\right] + n_y(\xi)\left[y - \frac{Y_i(\xi)}{W_i(\xi)}\right] \;=\; 0\,,$$

and this can be written as

$$n_x(\xi)\,x + n_y(\xi)\,y \;=\; h(\xi)\,,$$

where the support function $h(\xi)$, specifying the distance of the tangent line from the origin, is defined by

$$h(\xi) \;=\; \frac{n_x(\xi)X_i(\xi) + n_y(\xi)Y_i(\xi)}{W_i(\xi)}\,.$$

Substituting for $n_x(\xi), n_y(\xi), W_i(\xi), X_i(\xi), Y_i(\xi)$, we identify $f(\xi)$ and $g(\xi)$ as the numerator and denominator of the rational quartic function

$$h(\xi) \;=\; \frac{f(\xi)}{g(\xi)} \;=\; \frac{(\xi+1)^3(3-5\,\xi)}{3\,(\xi^4 - 8\,\xi^3 + 24\,\xi^2 - 20\,\xi + 5)}\,. \tag{32}$$

Finally, having identified $a(\xi), b(\xi)$ and $f(\xi), g(\xi)$ the expressions (23) can be evaluated. Upon factorizing them, we obtain

$$\begin{aligned}
W(\xi) &= 864\,\xi^3(\xi-1)^3[\,a^2(\xi)+b^2(\xi)\,]^3\,,\\
X(\xi) &= -\,864\,(\xi-1)^3[\,a^2(\xi)+b^2(\xi)\,]^3(\xi+1)(5\,\xi^2-4\,\xi+1)\,,\\
Y(\xi) &= -\,288\,(\xi-1)^2[\,a^2(\xi)+b^2(\xi)\,]^3(\xi+1)(13\,\xi^3-26\,\xi^2+20\,\xi-4)\,,
\end{aligned}$$

where

$$a^2(\xi)+b^2(\xi) \;=\; \xi^4 - 8\,\xi^3 + 24\,\xi^2 - 20\,\xi + 5\,. \tag{33}$$

Although W, X, Y are nominally of degree 18, they possess the common factor $288\,(\xi-1)^2[\,a^2(\xi)+b^2(\xi)\,]^3$ of degree 14 which, when divided out, yields the original quartic curve defined by (26).

Remark 4. The common factor $(\xi-1)^2[\,a^2(\xi)+b^2(\xi)\,]^3$ of $W(\xi), X(\xi), Y(\xi)$ in Example 8 is remarkable. If, instead of (31) and (32), we use freely–chosen quadratics $a(\xi), b(\xi)$ and quartics $f(\xi), g(\xi)$ in (23) then $W(\xi), X(\xi), Y(\xi)$ are of degree 18, and in general they possess no common factors. From (32) and (33), we see that $g(\xi) = 3\,[\,a^2(\xi)+b^2(\xi)\,]$ in Example 8, and it is evident from (22)–(23) that $W(\xi), X(\xi), Y(\xi)$ have $a^2(\xi)+b^2(\xi)$ as a common factor, but it is not obvious why its *cube* appears as a common factor.

The conditions on a,b and f,g that incur common factors among W,X,Y are not easily discerned from the expressions (23). Thus Example 8 highlights the ability of the direct integration approach, based on choosing polynomials u,v,w in (25), to generate low–degree curves without the need for a symbolic factorization to extract common factors among the expressions (23).

A similar situation arises with regard to the curve in Example 1 (for which $w(\xi)$ has simple roots). By the same reasoning used above, the polynomials $a(\xi),b(\xi)$ may be identified as

$$a(\xi) \;=\; \frac{\xi^3-3\,\xi^2+\xi-1}{\sqrt{2}}\,, \quad b(\xi) \;=\; \frac{-\,(\xi+1)(\xi-1)^2}{\sqrt{2}}\,,$$

and we obtain

$$h(\xi) \;=\; \frac{f(\xi)}{g(\xi)} \;=\; \frac{(\xi+1)^3(\xi^2-4\,\xi+1)(2\,\xi^2-5\,\xi+5)}{3\,(\xi^6-4\,\xi^5+5\,\xi^4-2\,\xi^3+3\,\xi^2-2\,\xi+1)}\,.$$

Substituting $a(\xi),b(\xi)$ and $f(\xi),g(\xi)$ into (22)–(23) and factorizing then gives

$$\begin{aligned}
W(\xi) &= -\,108\,\xi^3(\xi-1)^3(\xi^2-\xi+4)^3[\,a^2(\xi)+b^2(\xi)\,]^3\,,\\
X(\xi) &= -\,36\,\xi^2(\xi-1)^3(\xi^2-\xi+4)^3[\,a^2(\xi)+b^2(\xi)\,]^3(\xi+1)^2(\xi^2-5\,\xi+3)\,,\\
Y(\xi) &= -\,108\,\xi^3(\xi-1)^2(\xi^2-\xi+4)^3[\,a^2(\xi)+b^2(\xi)\,]^3(\xi+1)(\xi-2)^2\,,
\end{aligned}$$

where

$$a^2(\xi)+b^2(\xi) \;=\; \xi^6-4\,\xi^5+5\,\xi^4-2\,\xi^3+3\,\xi^2-2\,\xi+1\,.$$

W,X,Y are nominally of degree 30, 33, 32 but they have the common factor $36\,\xi^2(\xi-1)^2(\xi^2-\xi+4)^3[\,a^2(\xi)+b^2(\xi)\,]^3$ of degree 28, and eliminating it yields the original rational quintic curve (11). Again, this simplification is specific to the curve considered in Example 1 — substituting general cubics a,b and polynomials f and g of degree 7 and 6 in (22)–(23) yields an irreducible curve with $\deg(W)=30$, $\deg(X)=\deg(Y)=33$, and $\gcd(W,X,Y)=1$.

As in Example 8, we have $g=3\,(a^2+b^2)$ in this case, but it is not obvious why the cube of a^2+b^2 appears as a common factor of W,X,Y. Example 1 is thus another simple rational curve with a rational arc length, constructed by direct integration, that is not easy to identify from the expressions (22)–(23).

6 Generalization to space curves

We now briefly consider how the preceding discussion, which focused on plane rational curves with rational arc lengths, can be generalized to rational space curves. Consider the hodograph $\mathbf{r}'(\xi) = (x'(\xi), y'(\xi), z'(\xi))$ of a rational space curve, specified as

$$x'(\xi) \,=\, \frac{e(\xi)}{d(\xi)}\,, \quad y'(\xi) \,=\, \frac{f(\xi)}{d(\xi)}\,, \quad z'(\xi) \,=\, \frac{g(\xi)}{d(\xi)}\,,$$

for polynomials $d(\xi), e(\xi), f(\xi), g(\xi)$ with at least one of $\gcd(e, d)$, $\gcd(f, d)$, $\gcd(g, d)$ equal to 1. As in the planar case, the denominator polynomial $d(\xi)$ cannot have any simple roots if $x(\xi), y(\xi), z(\xi)$ are to be rational, and we choose $d(\xi) = w^2(\xi)$, where $w(\xi)$ is a polynomial with only simple roots.

Now for the parametric speed of $\mathbf{r}(\xi)$ to be rational, $e^2(\xi) + f^2(\xi) + g^2(\xi)$ must be a perfect square, and this implies [1] that

$$\begin{aligned}
e(\xi) &= u^2(\xi) + v^2(\xi) - p^2(\xi) - q^2(\xi)\,, \\
f(\xi) &= 2\,[\,u(\xi)q(\xi) + v(\xi)p(\xi)\,]\,, \\
g(\xi) &= 2\,[\,v(\xi)q(\xi) - u(\xi)p(\xi)\,]\,,
\end{aligned}$$

for polynomials $u(\xi), v(\xi), p(\xi), q(\xi)$ and consequently $e^2(\xi)+f^2(\xi)+g^2(\xi) = [\,u^2(\xi)+v^2(\xi)+p^2(\xi)+q^2(\xi)\,]^2$. To obtain a rational space curve with rational arc length, we must choose $u(\xi), v(\xi), p(\xi), q(\xi)$, for a given $w(\xi)$ with simple roots, so as to ensure that the indefinite integrals of the rational functions

$$x' \,=\, \frac{u^2 + v^2 - p^2 - q^2}{w^2}\,, \quad y' \,=\, \frac{2\,(uq + vp)}{w^2}\,, \quad z' \,=\, \frac{2\,(vq - up)}{w^2}\,,$$

$$s' \,=\, \frac{u^2 + v^2 + p^2 + q^2}{w^2}\,,$$

are rational expressions in ξ. A sufficient condition for this can be formulated through a simple adaptation of the approach used for planar curves. Namely, for a given $w(\xi)$ we choose right–hand sides in (16) so as to obtain solutions defining four linearly independent and pairwise relatively prime polynomials $u(\xi), v(\xi), p(\xi), q(\xi)$ such that u^2/w^2, v^2/w^2, p^2/w^2, q^2/w^2 all have rational integrals, and this ensures that $uq/w^2, vp/w^2, vq/w^2, up/w^2$ also have rational integrals. Hence, integration of x', y', z', s' yields a rational space curve with a rational arc length.

Example 9. For $w(\xi) = \xi(\xi - 1)$, the relatively prime polynomials

$$u(\xi) \,=\, -\,\xi^2 + \xi - 1\,, \quad v(\xi) \,=\, \xi^3 - 2\,\xi + 2\,,$$

$$p(\xi) \,=\, -\,2\,\xi^3 + 3\,\xi^2 + \xi - 1\,, \quad q(\xi) \,=\, -\,\xi^3 + 4\,\xi^2 - 2\,\xi + 2\,,$$

correspond respectively to values $P = (1,1)^T$, $P = (-1,-2)^T$, $P = (-1,1)^T$, $P = (-3,-2)^T$ and $R = (1,-1)^T$, $R = (-2,1)^T$, $R = (1,1)^T$, $R = (-2,3)^T$ in equation (16). These polynomials generate the rational quintic curve $\mathbf{r}(\xi)$ specified by the homogeneous coordinates

$$\begin{aligned}
W(\xi) &= 3\,\xi(\xi-1)\,, \\
X(\xi) &= -\,4\,\xi^5 + 22\,\xi^4 - 18\,\xi^3 - 10\,\xi^2 + 34\,\xi\,, \\
Y(\xi) &= -\,4\,\xi^5 + 4\,\xi^4 + 12\,\xi^3 + 26\,\xi^2 - 2\,\xi - 24\,, \\
Z(\xi) &= -\,2\,\xi^5 + 2\,\xi^4 + 36\,\xi^3 - 32\,\xi^2 - 46\,\xi + 18\,,
\end{aligned}$$

which has the rational arc length function

$$s(\xi) \,=\, \frac{6\,\xi^5 - 18\,\xi^4 + 12\,\xi^3 - 30\,\xi^2 - 36\,\xi + 30}{3\,\xi(\xi-1)}\,,$$

where we assume integration constants such that $\mathbf{r}(-1) = \mathbf{0}$ and $s(-1) = 0$.

7 Closure

A novel approach to characterizing rational curves with rational arc length functions has been introduced. For planar curves this is based on identifying, for a given polynomial $w(\xi)$ with simple roots, conditions on two polynomials $u(\xi), v(\xi)$ satisfying $\gcd(u,v) = \gcd(u,w) = \gcd(v,w) = 1$ which ensure that the integrals of u^2/w^2, uv/w^2, v^2/w^2 are rational functions. A generalization to the case of rational space curves was also briefly sketched.

The focus of this study was on elucidating the underlying basic theory to support the construction of rational curves with rational arc lengths by direct integration. This facilitates the development of algorithms for the design of curve segments that satisfy desired geometrical constraints (e.g., prescribed end points and tangents), as required in practical applications.

The direct integration scheme was compared with prior theory concerning rational curves with rational arc lengths, based upon the dual representation for rational Pythagorean–hodograph curves, and examples were used to show that straightforward constuctions of low–degree curves are possible in cases where the prior approach requires symbolic factorization of the homogeneous coordinates to identify and extract common non–constant factors.

A number of issues concerning the methodology proposed herein deserve further investigation. First, as noted above, it should be exploited to develop algorithms that address practical design problems — e.g., the construction of G^1 Hermite interpolants with prescribed arc lengths. Second, the theory should be extended to relax the assumptions made herein that $w(\xi)$ has only simple roots, and that $\gcd(u,v) = \gcd(u,w) = \gcd(v,w) = 1$. Finally, the non–obvious circumstances that incur common factors in the homogeneous coordinates (23) deserve further elucidation. These are non–trivial problems, that will require separate substantive studies.

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